

Zero-Forcing Precoding for Frequency Selective MIMO Channels with H^∞ Criterion and Causality Constraint ^{*}

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Abstract

We consider zero-forcing equalization of frequency selective MIMO channels by causal and linear time-invariant precoders in the presence of intersymbol interference. Our motivation is twofold. First, we are concerned with the optimal performance of causal precoders from a worst case point of view. Therefore we construct an optimal causal precoder, whereas contrary to other works our construction is not limited to finite or rational impulse responses. Moreover we derive a novel numerical approach to computation of the optimal performance index achievable by causal precoders for given channels. This quantity is important in the numerical determination of optimal precoders.

Key words: MIMO, Intersymbol Interference, Filterbank, Precoder, Equalizer, Causality, Bezout Identity, Matrix Corona Problem, Minimum Norm

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1 Introduction

Many of today's state-of-the-art wireless systems adopt multiple-input multiple-output (MIMO) transmission to increase spectral efficiency together with multi-carrier methods to cope with intersymbol interference (ISI). Multi-carrier methods simplify channel equalization because they decompose frequency selective channels into multiple flat fading channels (the so called carriers), which can be easily equalized. While multi-carrier transmission offers many advantages including effective channel equalization, it also exhibits some drawbacks regarding the peak-to-average power ratio (PAPR). Often single-carrier transmission, where the frequency selective channel is approached directly, is considered as an alternative to multi-carrier transmission [1,2,3]. While therefore single-carrier transmission is interesting on its own, it has been further shown in [4,5], that in fact many common multi-carrier, code-multiplex and space-time block-code systems can be modeled as single-carrier systems by virtual enhancement of the MIMO system. Various authors used this approach to derive new equalization methods based on single-carrier equalization in order to exploit joint equalization of spatial, time and code or frequency domains [4,5,6,7]. There, and generally for linear time-invariant (LTI) equalization of single-carrier systems with zero-forcing and causality constraint, one usually solves the so-called *Bezout Identity*

$$H(e^{i\theta})G(e^{i\theta}) = I \quad (0 \leq \theta < 2\pi),$$

where the matrix-valued transfer function H of a stable and causal LTI system (the frequency selective MIMO channel) is given, and a transfer function G of a stable and causal LTI precoder, which equalizes H , has to be computed [8]. Transmitters may use such G to pre-equalize the channel. Alternatively, receivers can also solve the Bezout Identity for the transposed channel H^T (i.e. $H^T G = I$) and equalize the channel H with the transposed solution G^T . The main difficulty in solving the Bezout Identity is the causality of G , because the naive approach

$$G(e^{i\theta}) = H(e^{i\theta})^* [H(e^{i\theta})H(e^{i\theta})^*]^{-1} \quad (0 \leq \theta < 2\pi)$$

of a *pseudoinverse* generally results in a non-causal precoder [9]. If the number of channel inputs equals the number of channel outputs, the pseudoinverse is the unique solution to the Bezout Identity. The situation changes if the number of inputs of H is larger than the number of outputs. Now precoders for H no longer have to be unique. Usually this non-uniqueness then is exploited to choose a causal G that is optimal in some sense. The two common optimality conditions are *minimality of the equalizers energy* and *minimality of the equalizers peak value*, respectively. The minimal energy condition corresponds to the classical approach of signal-to-noise ratio (SNR) maximization [5,7,10]. However, this approach is only feasible if the statistical properties of the noise

are known. For unknown noise statistics, it cannot be applied. Picking up an idea from robust control (see e.g. [11]), where one is concerned with unpredictable errors that arise e.g. due to uncertain modeling, instead minimization of the equalizers peak value has been proposed [6]. As we shall see later, this minimizes the worst case error instead of the average error, which cannot be determined due to the unknown noise statistics.

In this paper we are interested in the optimal performance that causal precoders and equalizers can archive regarding the worst case error. Therefore we show how a solution to the Bezout Identity with minimal peak value can be constructed. We discuss why this gives the best upper bounds on various perturbations in the system. Contrary to other ways to solve the Bezout Identity, our construction holds for the most general case of systems with infinite impulse responses (which are not required to be rational) and even infinite input and output vectors, i.e. we allow systems to have infinite temporal as well as infinite spatial dimension. We further give a new result on the numerical computation of the minimum peak value achievable by causal solutions to the Bezout Identity if the numbers of inputs and outputs are finite. This is important because for all methods known to the authors that solve the Bezout Identity with minimal peak value in a numerically exploitable way, the minimal peak value has to be known in advance [6,12]. Therefore efficient computation of the minimal peak value is important for numerical solution of the Bezout Identity. We point out that the optimization approach in [13] requires no prior knowledge of the minimal peak value. However, it only computes finite impulse response solutions to the Bezout Identity, which are generally suboptimal.

We proceed as follows. In Section 2 we give our problem statement after we have introduced some notation and necessary basic mathematical concepts. We further discuss the practical interpretation of our problem statement. In Section 3 we derive our results on the numerical computation of the minimal peak value achievable by causal solutions to the Bezout Identity. Then a optimal causal precoder is constructed in Section 4. We finally draw conclusions in Section 5.

2 Preliminaries

2.1 Notation

We denote the complex numbers by \mathbb{C} , the complex matrices with m rows and n columns by $\mathbb{C}^{m \times n}$ and the complex column vectors by $\mathbb{C}^m := \mathbb{C}^{m \times 1}$. The complex unit disc is given as $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, its border is the unit circle

$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Complex conjugation is denoted by $(\bar{\cdot})$, taking adjoints in a Hilbert space by $(\cdot)^*$. Furthermore \mathcal{H}, \mathcal{E} and \mathcal{E}_* denote separable Hilbert spaces with scalar products $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{E}_*}$, respectively. By $\mathcal{H} \oplus \mathcal{E}$ we mean the direct Hilbert sum, i.e. the space $\mathcal{H} \times \mathcal{E}$ equipped with scalar product $\langle h \oplus e, g \oplus f \rangle_{\mathcal{H} \oplus \mathcal{E}} := \langle h, g \rangle_{\mathcal{H}} + \langle e, f \rangle_{\mathcal{E}}$. The space of bounded linear operators between \mathcal{E} and \mathcal{E}_* is denoted by $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$. It is equipped with the operator norm $\|T\|_{\text{op}} := \sup_{e \in \mathcal{E}, \|e\|_{\mathcal{E}}=1} \|Te\|_{\mathcal{E}_*}$. On any space the identity operator is written as I . For matrices $A \in \mathbb{C}^{m \times n}$ the smallest and largest singular value will be denoted by $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$, respectively. The closure of a set M is denoted by $\text{closure } M$, the space spanned by all linear combinations of its elements by $\text{span } M$.

As usual, $L_{\mathbb{T}}^p(X)$ denotes the space of (equivalence classes of) p -integrable functions on \mathbb{T} with values in a Banach space X . The norm in $L_{\mathbb{T}}^p(X)$ is $\|f\|_p^p := \int_{\theta=0}^{2\pi} \|f(e^{i\theta})\|_X^p \frac{d\theta}{2\pi}$ for $1 \leq p < \infty$ and $\|f\|_{\infty} := \text{esssup}_{\zeta \in \mathbb{T}} \|f(\zeta)\|_X = \inf \{m > 0 : \mu(\{\zeta \in \mathbb{T} : \|f(\zeta)\|_X > m\}) = 0\}$ for $p = \infty$, where μ denotes the Lebesgue measure. We refer to [14, Section 3.11] and the references therein for details on integration of vector- and operator-valued functions. If $p = 2$, $L_{\mathbb{T}}^2(\mathcal{E})$ equipped with the scalar product $\langle f, g \rangle_2 := \int_{\theta=0}^{2\pi} \langle f(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathcal{E}} \frac{d\theta}{2\pi}$ is a Hilbert space. For $F \in L_{\mathbb{T}}^{\infty}(\mathcal{L}(\mathcal{E}, \mathcal{E}_*))$ we denote the point-wise adjoint by F^* , i.e. $F^*(\zeta) = (F(\zeta))^*$ almost everywhere on the unit circle.

2.2 Basic Results and Concepts

2.2.1 Hardy Spaces and Toeplitz Operators

We introduce the usual *Hardy spaces on the disc* by

$$H_{\mathbb{D}}^2(\mathcal{E}) := \left\{ u : \mathbb{D} \rightarrow \mathcal{E} : u \text{ analytic}, \|u\|_2^2 := \sup_{0 < r < 1} \int_{\theta=0}^{2\pi} \|u(re^{i\theta})\|_{\mathcal{E}}^2 \frac{d\theta}{2\pi} < \infty \right\},$$

$$H_{\mathbb{D}}^{\infty}(\mathcal{E}, \mathcal{E}_*) := \left\{ F : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{E}_*) : F \text{ analytic}, \|F\|_{\infty} := \sup_{z \in \mathbb{D}} \|F(z)\|_{\text{op}} < \infty \right\}.$$

The Hardy spaces play an important role in systems theory, since they are the set of transfer functions of causal finite energy signals and causal and energy-stable transfer functions for LTI systems, respectively [15]. Definition is also possible on the upper half plane instead of the unit disc. On both domains, the Hardy functions are completely determined by their values on the borders of the domain. Therefore, each Hardy function on the unit disc has a corresponding function on the circle. The space of those corresponding functions can be given as follows.

For functions $f \in L^1_{\mathbb{T}}(\mathcal{E})$ or $f \in L^1_{\mathbb{T}}(\mathcal{L}(\mathcal{E}, \mathcal{E}_*))$ the k -th *Fourier coefficient* is

$$\hat{f}_k := \int_{\theta=0}^{2\pi} f(e^{i\theta}) e^{-ik\theta} \frac{d\theta}{2\pi} \quad (k \in \mathbb{Z}).$$

Therewith, the *Hardy spaces on the circle* are given by

$$\begin{aligned} H^2_{\mathbb{T}}(\mathcal{E}) &:= \left\{ u \in L^2_{\mathbb{T}}(\mathcal{E}) : \hat{u}_k = 0 \text{ for } k < 0 \right\}, \\ H^\infty_{\mathbb{T}}(\mathcal{E}, \mathcal{E}_*) &:= \left\{ F \in L^\infty_{\mathbb{T}}(\mathcal{L}(\mathcal{E}, \mathcal{E}_*)) : \hat{F}_k = 0 \text{ for } k < 0 \right\}. \end{aligned}$$

It is important to know that the two notions of Hardy spaces on disc and circle are equivalent, since by Fatou's Theorem the radial limit $(bu)(e^{i\theta}) := \lim_{r \nearrow 1} u(re^{i\theta})$ exists almost everywhere and the mapping b is an isometric isometry between the Hardy spaces on disc and circle (see [14, Th. 3.11.7, 3.11.10]). Therefore we will only explicitly distinguish between those spaces if necessary, and simply write $H^2(\mathcal{E})$ and $H^\infty(\mathcal{E}, \mathcal{E}_*)$ otherwise.

An important property of $L^2_{\mathbb{T}}(\mathcal{E})$ is *Parseval's Relation* ([16, p. 184]), by which

$$\|u\|_2^2 = \sum_{k=-\infty}^{\infty} \|\hat{u}_k\|_{\mathcal{E}}^2 \text{ for all } u \in L^2_{\mathbb{T}}(\mathcal{E}).$$

We will now introduce *Toeplitz operators*, which are the standard example for operators on Hardy spaces and which will also play an important role in what follows. The orthogonal projection $(P_+u)(\zeta) := \sum_{k=0}^{\infty} \hat{u}_k \zeta^k$ from $L^2_{\mathbb{T}}(\mathcal{E})$ into $H^2_{\mathbb{T}}(\mathcal{E})$ is called the *Riesz Projection*. The projection from $H^2_{\mathbb{T}}(\mathcal{E})$ into the space of degree N polynomials is $(P_Nu)(\zeta) := \sum_{k=0}^N \hat{u}_k \zeta^k$. Now for $F \in L^\infty_{\mathbb{T}}(\mathcal{E}, \mathcal{E}_*)$ the *Toeplitz operator with symbol F* is the operator which maps $H^2_{\mathbb{T}}(\mathcal{E})$ into $H^2_{\mathbb{T}}(\mathcal{E}_*)$ via $T_F u := P_+(Fu)$.

The next result allows us to get an exact estimate of the minimum norm achievable by solutions of the Bezout Identity.

Theorem 1 ([14, Th. 9.2.1]) *Let $F \in H^\infty(\mathcal{E}, \mathcal{E}_*)$ and $\delta > 0$. Then some $G \in H^\infty(\mathcal{E}_*, \mathcal{E})$ with $\|G\|_\infty \leq \delta^{-1}$ and $F(z)G(z) = I$ for all $z \in \mathbb{D}$ exists if and only if*

$$\|T_{F^*}u\|_2 \geq \delta \|u\|_2 \text{ for all } u \in H^2(\mathcal{E}_*).$$

2.2.2 Schur Class

Functions in the unit ball of $H^\infty(\mathcal{E}, \mathcal{E}_*)$, the so-called *Schur class*

$$S(\mathcal{E}, \mathcal{E}_*) := \{F \in H^\infty(\mathcal{E}, \mathcal{E}_*) : \|F\|_\infty \leq 1\},$$

have some special properties, which will turn out to be useful in the construction of a minimum norm right inverse. Every Schur function can be factorized as follows.

Theorem 2 ([17, Th. 2.1]) *Let $F : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$. Then $F \in S(\mathcal{E}, \mathcal{E}_*)$ if and only if there exists a holomorphic function $W : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{E}_*)$ such that*

$$I - F(z)F(w)^* = (1 - z\bar{w})W(z)W(w)^* \quad (z, w \in \mathbb{D}).$$

Note that W can be given explicitly, see [17, Sec. 3.3]. We finish with the observation that also certain block operators define Schur functions.

Lemma 3 ([12, Lem. 2]) *Let $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{E}, \mathcal{H} \oplus \mathcal{E}_*)$ with $\|T\|_{\text{op}} \leq 1$. Then T has a unique block matrix representation*

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

and the function

$$F : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{E}_*), \quad F(z) := D + Cz(I - zA)^{-1}B$$

is Schur, i.e. $F \in S(\mathcal{E}, \mathcal{E}_*)$.

Functions defined as F in the Lemma above are known in operator theory as characteristic functions, while unitary operators like T are known as unitary colligations. Those concepts resemble much the concept of a transfer function and a state-space realization in control theory. We refer to [17,18] for details.

2.3 Problem Formulation

Before we give an exact problem formulation we introduce and discuss the target objective

$$\gamma_{\text{opt}}(H) := \inf (\{ \|G\|_{\infty} : G \in H^{\infty}(\mathcal{E}_*, \mathcal{E}), H(z)G(z) = I \text{ for all } z \in \mathbb{D} \} \cup \{\infty\}),$$

which is, as we will see, a tight lower bound on the worst-case transmit energy enhancement of causal precoders for the channel H , and a measure for the achievable robustness against imperfectly known channel transfer functions. Note that in particular $\gamma_{\text{opt}}(H) = \infty$ if and only if H has no right inverse in H^{∞} . We always assume $H \in H^{\infty}(\mathcal{E}, \mathcal{E}_*)$ unless we explicitly mention otherwise.

It was shown in [19] that if $\dim \mathcal{E}_* < \infty$, existence of a right inverse in H^{∞} is

further equivalent to

$$H(z)H(z)^* \geq \delta^2 I \text{ for some } \delta > 0 \text{ and all } z \in \mathbb{D}.$$

It is somewhat surprising that although by the result from [19] $\gamma_{\text{opt}} < \infty$ if and only if

$$\delta_c := \sup \left\{ \delta \geq 0 \mid H(z)H(z)^* \geq \delta^2 I \text{ for all } z \in \mathbb{D} \right\} > 0,$$

δ_c has no direct connection to γ_{opt} , i.e. γ_{opt} cannot be computed from δ_c [9]. However, as we will see, it is important to know γ_{opt} in advance of the construction of an optimal precoder. Therefore we derive a new method for numerical computation of γ_{opt} and then solve the following problem.

Problem 4 *Let $\gamma_{\text{opt}}(H) < \infty$. How can $G \in H^\infty(\mathcal{E}_*, \mathcal{E})$ with $H(z)G(z) = I$ for all $z \in \mathbb{D}$ and $\|G\|_\infty = \gamma_{\text{opt}}(H)$ be constructed?*

We close this section with a short discussion in which sense minimization of the infinity norm in Problem 4 gives optimal filters. The input-output relation of a frequency selective MIMO channel is given by

$$y(\zeta) = H(\zeta)x(\zeta) + n(\zeta) \quad (\zeta \in \mathbb{T}),$$

where H denotes the channel, x the transmitted signals and y and n the received signals and additive noise, respectively. If a precoder G for H is used to pre-distort the transmitted signals, this input-output relation changes to

$$y(\zeta) = H(\zeta)G(\zeta)x(\zeta) + n(\zeta) = x(\zeta) + n(\zeta) \quad (\zeta \in \mathbb{T}).$$

There are two advantages in minimizing the infinity norm of the precoder.

The first advantage is minimization of the transmit signals energy. The energy necessary to transmit a signal x using the precoder G is given by $\|Gx\|_2^2$. Without loss of generality, let us assume that $\|x\|_2^2 = 1$. Then, it can be shown that the transmit energy necessary in the worst case is exactly $\|G\|_\infty^2$, i.e.

$$\sup_{x \in H^2(\mathcal{E}_*), \|x\|_2=1} \|Gx\|_2^2 = \|G\|_\infty^2.$$

Thus, minimizing $\|G\|_\infty$ guarantees the lowest amount of necessary transmit energy. If equalizers instead of precoders are considered, i.e.

$$y(\zeta) = G(\zeta)[H(\zeta)x(\zeta) + n(\zeta)] = x(\zeta) + G(\zeta)n(\zeta) \quad (\zeta \in \mathbb{T}),$$

this is equivalent to minimal worst case noise enhancement.

The second advantage of minimization of the infinity norm is robustness. Assume an imperfectly known channel transfer function $H_\Delta = H + \Delta$ with right inverse G_Δ , where H is the correct channel and Δ is a perturbation. Using

the same argument as before, we see that the energy of the worst error that can result from the perturbation equals

$$\sup_{x \in H^2(\mathcal{E}_*), \|x\|_2=1} \|x - HG_\Delta x\|_2^2 = \sup_{x \in H^2(\mathcal{E}_*), \|x\|_2=1} \|\Delta G_\Delta x\|_2^2 = \|\Delta G_\Delta\|_\infty^2.$$

Since it holds $\|\Delta G_\Delta\|_\infty^2 \leq \|\Delta\|_\infty^2 \|G_\Delta\|_\infty^2$, and this inequality can become sharp e.g. for $\Delta = \delta I$, we see that minimizing $\|G_\Delta\|_\infty$ also minimizes the worst case error that results from an imperfectly known channel transfer function. This argument applies to equalizers in the same way it applies to precoders.

3 Computation of the Optimal Norm

This section deals with the computation of the optimal norm γ_{opt} achievable by solutions to the Bezout Identity. Since many algorithms which directly solve Problem 4 only compute suboptimal solutions, i.e. given $\gamma > \gamma_{\text{opt}}$ they compute a right inverse G_γ with norm $\|G_\gamma\|_\infty < \gamma$, it is important to know the optimal value for γ in advance [6,12]. We point out that computation of γ_{opt} also arises in other contexts, see e.g. Remark 1 in [20] (with the next corollary in mind).

We start with an exact (but incomputable) formula for γ_{opt} . The next two corollaries are direct consequences of Theorem 1.

Corollary 5 *For $\rho(H) := \inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*}u\|_2$, it holds $\gamma_{\text{opt}}(H) = \rho(H)^{-1}$.*

Corollary 6 *If $\gamma_{\text{opt}}(H) < \infty$, a right inverse $G \in H^\infty(\mathcal{E}_*, \mathcal{E})$ with $\|G\|_\infty = \gamma_{\text{opt}}(H)$ exists.*

The interesting thing about Corollary 5 is that it shows us why the optimal causal equalizer cannot perform better than the optimal non-causal one. Note that the optimal norm for non-causal equalizers is given by

$$\left(\inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|H^*u\|_2 \right)^{-1}$$

(see [9]), which is the same formula as Corollary 5, except for the additional Riesz projection P_+ :

$$\gamma_{\text{opt}}(H) = \left(\inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|P_+(H^*u)\|_2 \right)^{-1}.$$

It is now clear that causal equalizers perform worse because the signal energy of u which is mapped into the non-causal part of H^*u is cut off. How much

energy is shifted into the non-causal part thereby depends on the Fourier coefficients of H^* , which are related to H by $\widehat{H^*}_k = \hat{H}^*_{-k}$ for $k \in \mathbb{Z}$.

We now derive a computable approximation of γ_{opt} . The main idea will be to approximate the relation $\gamma_{\text{opt}} = \rho^{-1}$ from Corollary 5. In order to compute γ_{opt} , we try to approximate ρ with

$$\rho_N(H) := \inf_{u \in P_N H^2(\mathcal{E}_*), \|u\|_2=1} \|P_N T_{H^*} u\|_2,$$

i.e. we restrict domain and image of T_{H^*} to polynomials of degree N and take the infimum for this restriction. Because $P_N T_{H^*} P_N$ is linear and finite dimensional, it can be represented by a matrix.

The main result of this section is the following.

Theorem 7 *The sequence $\{\rho_N(H)\}_N$ is monotonically decreasing and converges with limit*

$$\lim_{N \rightarrow \infty} \rho_N(H) = \rho(H) = \gamma_{\text{opt}}(H)^{-1}.$$

If $H \in H^\infty(\mathbb{C}^{m \times n})$ with $m \leq n$,¹ and

$$\Gamma_{H,N} := \begin{bmatrix} \hat{H}_0^* & \hat{H}_1^* & \dots & \hat{H}_N^* \\ 0 & \hat{H}_0^* & \dots & \hat{H}_{N-1}^* \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \hat{H}_0^* \end{bmatrix} \in \mathbb{C}^{n(N+1) \times m(N+1)},$$

ρ_N can be computed as $\rho_N(H) = \sigma_{\min}(\Gamma_{H,N})$.

PROOF. We only sketch the proof here, the full proof is given in the appendix. It consists of three main steps. The first step is to show that the sequence $\{\rho_N(H)\}_N$ is monotonically decreasing and lower bounded by $\rho(H)$. The main idea is that the relation

$$\rho_N(H) = \inf_{u \in P_N H^2(\mathcal{E}_*), \|u\|_2=1} \|P_N T_{H^*} u\|_2 = \inf_{u \in P_N H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2$$

holds for every $N \in \mathbb{N}$ and thus the infimum is always taken over the same target objective, but over a space which increases with N . This is done in the appendix in Proposition 14. In a second step it is shown that the lower bound $\rho(H)$ for $\{\rho_N(H)\}_N$ is sharp. Therefore for arbitrary $\epsilon > 0$ a sequence $\{u_N\}_N$ such that

$$u_N \in P_N H^2(\mathcal{E}_*), \|u_N\|_2 = 1 \text{ and } \lim_{N \rightarrow \infty} \|P_N T_{H^*} u_N\|_2 \leq \rho(H) + \epsilon$$

¹ Note that trivially $\gamma_{\text{opt}}(H) = \infty$ for $m > n$.

is constructed in the appendix in Proposition 15. Thus $\rho_N(H)$ converges to $\rho(H)$, which is equal to $\gamma_{\text{opt}}(H)^{-1}$ by Corollary 5. Finally Proposition 16 in the appendix gives the formula for computation of $\rho_N(H)$ via singular value decomposition if H is matrix-valued. \blacksquare

Since the arguments used to prove Theorem 7 hold analogously if we approximate

$$\sup_{u \in H^2(\mathbb{C}^m), \|u\|_2=1} \|T_{H^*}u\|_2 = \|T_{H^*}\|_{\text{op}} = \|T_H^*\|_{\text{op}} = \|T_H\|_{\text{op}} = \|H\|_{\infty}$$

instead of $\rho_N(H) = \inf_{u \in H^2(\mathbb{C}^m), \|u\|_2=1} \|T_{H^*}u\|_2$, we also see that for $H \in H^{\infty}(\mathbb{C}^{m \times n})$ the sequence $\{\sigma_{\max}(\Gamma_{H,N})\}_N$ is monotonically increasing and converges with limit

$$\lim_{N \rightarrow \infty} \sigma_{\max}(\Gamma_{H,N}) = \|H\|_{\infty}.$$

We note that the well-known fact that the limit $\|H\|_{\infty}$ of $\sigma_{\max}(\Gamma_{H,N})$ can be found by performing a grid search over all frequencies, i.e.

$$\lim_{N \rightarrow \infty} \sigma_{\max}(\Gamma_{H,N}) = \|H\|_{\infty} = \text{esssup}_{\zeta \in \mathbb{T}} \sigma_{\max}(H(\zeta)),$$

does not carry over to computation of $\gamma_{\text{opt}}(H)$. Here, in general we have

$$\lim_{N \rightarrow \infty} \sigma_{\min}(\Gamma_{H,N}) = \gamma_{\text{opt}}(H)^{-1} \neq \text{essinf}_{\zeta \in \mathbb{T}} \sigma_{\min}(H(\zeta)).$$

This dichotomy results from the fact that while indeed

$$\sup_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|H^*u\|_2 = \sup_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|P_+(H^*u)\|_2,$$

in general we have

$$\inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|H^*u\|_2 \neq \inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|P_+(H^*u)\|_2.$$

This can be easily seen in the next example.

Example 8 Set $H(\zeta) = \zeta$ for $\zeta \in \mathbb{T}$. Then by Parseval's Relation

$$\inf_{u \in H^2(\mathbb{C}), \|u\|_2=1} \|H^*u\|_2 = \inf_{u \in H^2(\mathbb{C}), \|u\|_2=1} \|u\|_2 = 1,$$

however for $u(z) = 1$ we have $(H^*u)(\zeta) = \bar{\zeta}$ and therefore

$$\|P_+(H^*u)\|_2 = \|0\|_2 = 0.$$

It is also important to note that Theorem 7 does not generalize to the case $H \in L_{\mathbb{T}}^{\infty}$. We give an example where $\rho(H) = 1$, an inverse in H^{∞} exists, but the smallest singular values of the finite sections do not converge to $\rho(H)$.

Example 9 Set $H(\zeta) := \bar{\zeta}$ for $\zeta \in \mathbb{T}$. Then by Parseval's Relation

$$\rho(H) = \inf_{u \in H^2(\mathbb{C}), \|u\|_2=1} \|T_{H^*}u\|_2 = \inf_{u \in H^2(\mathbb{C}), \|u\|_2=1} \|T_{\zeta}u\|_2 = \inf_{u \in H^2(\mathbb{C}), \|u\|_2=1} \|u\|_2 = 1.$$

Further, H has a inverse in H^∞ , i.e. $G(\zeta) = \zeta$. However,

$$\sigma_{\min} \left(\begin{bmatrix} \hat{H}_0^* & \hat{H}_1^* & \dots & \hat{H}_N^* \\ \hat{H}_{-1}^* & \hat{H}_0^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{H}_1^* \\ \hat{H}_{-N}^* & \dots & \hat{H}_{-1}^* & \hat{H}_0^* \end{bmatrix} \right) = \sigma_{\min} \left(\begin{bmatrix} 0 & \dots & \dots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix} \right) = 0$$

for all $N \in \mathbb{N}$.

4 Construction of the Optimal Causal Precoder

In this section we construct a minimum norm solution to the Bezout Identity, i.e. we solve Problem 4. The major idea of the proof is the following. We first show how to construct right inverses with norm at most one. Then given any $H \in H^\infty(\mathcal{E}, \mathcal{E}_*)$, we apply this technique to the scaled function $\gamma_{\text{opt}} H$. Appropriate rescaling of the obtained inverse will result in a minimum norm right inverse.

4.1 Schur Right Inverse

The first step is construction of a Schur right inverse. Therefore we factorize the function to be inverted similar to Theorem 2 and use this factorization to construct a contraction of the form of T in Lemma 3. The characteristic function of this contraction then is the wanted right inverse. This is a variant of the technique known as “lurking isometry method”, which has been introduced by Ball and Trent [17, Th. 5.2] and independently Agler and McCarthy [21] to solve the Bezout Identity.

We start with the factorization.

Lemma 10 Let H have a right inverse $G \in S(\mathcal{E}_*, \mathcal{E})$. Then there exists a holomorphic function $W : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{E}_*)$ such that

$$H(z)H(w)^* - I = (1 - z\bar{w})W(z)W(w)^* \quad (z, w \in \mathbb{D}). \quad (1)$$

PROOF. By Theorem 2 there exists a holomorphic function $\tilde{W} : \mathbb{D} \rightarrow \mathcal{L}(\tilde{\mathcal{H}}, \mathcal{E})$ such that $I - G(z)G(w)^* = (1 - z\bar{w})\tilde{W}(z)\tilde{W}(w)^*$. Thus

$$H(z)H(w)^* - H(z)G(z)G(w)^*H(w)^* = (1 - z\bar{w})H(z)\tilde{W}(z)\tilde{W}(w)^*H(w)^*.$$

Since $HG = I$ we obtain with $W(z) := H(z)\tilde{W}(z)$ that

$$H(z)H(w)^* - I = (1 - z\bar{w})W(z)W(w)^*.$$

■

We can now introduce the appropriate block operator.

Definition 11 *Let H have a decomposition like (1) in Lemma 10. We define the sets*

$$D_0 := \text{closure} \left(\text{span} \left\{ \begin{bmatrix} \bar{w}W(w)^* \\ H(w)^* \end{bmatrix} e_* : w \in \mathbb{D}, e_* \in \mathcal{E}_* \right\} \right) \subset \mathcal{H} \oplus \mathcal{E},$$

$$R_0 := \text{closure} \left(\text{span} \left\{ \begin{bmatrix} W(w)^* \\ I \end{bmatrix} e_* : w \in \mathbb{D}, e_* \in \mathcal{E}_* \right\} \right) \subset \mathcal{H} \oplus \mathcal{E}_*,$$

and a function $V_0 : D_0 \rightarrow R_0$ by

$$\sum_{k=0}^{\infty} c_k \begin{bmatrix} \bar{w}W(w)^* \\ H(w)^* \end{bmatrix} e_{*k} \mapsto \sum_{k=0}^{\infty} c_k \begin{bmatrix} W(w)^* \\ I \end{bmatrix} e_{*k}.$$

Note that it can be easily shown with (1) that V_0 is a isometry, i.e.

$$\left\langle V_0 \begin{bmatrix} h \\ e \end{bmatrix}, V_0 \begin{bmatrix} h \\ e \end{bmatrix} \right\rangle_{\mathcal{H} \oplus \mathcal{E}_*} = \left\langle \begin{bmatrix} h \\ e \end{bmatrix}, \begin{bmatrix} h \\ e \end{bmatrix} \right\rangle_{\mathcal{H} \oplus \mathcal{E}} \quad \text{for all } \begin{bmatrix} h \\ e \end{bmatrix} \in \mathcal{H} \oplus \mathcal{E}.$$

Later we will use this fact when we apply Lemma 3 to an extension of V_0 . The wanted right inverse can now be given explicitly.

Theorem 12 *Let H have a decomposition like (1) in Lemma 10 and construct V_0 as in Definition 11. Denote by*

$$V_{00} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}$$

the continuation of V_0 with zero, i.e.

$$V_{00}d = \begin{cases} V_0d, & d \in D_0 \\ 0, & d \notin D_0 \end{cases}.$$

Then the function

$$G(z) := D^* + B^*(I - zA^*)^{-1}zC^* \quad (z \in \mathbb{D})$$

is a Schur right inverse of H , i.e. $G \in S(\mathcal{E}_*, \mathcal{E})$ and $HG = I$.

PROOF. Let $w \in \mathbb{D}$. By construction of V_{00} it holds

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{w}W(w)^* \\ H(w)^* \end{bmatrix} e_* = \begin{bmatrix} W(w)^* \\ I \end{bmatrix} e_*,$$

for all $e_* \in \mathcal{E}_*$, which is equivalent to

$$A\bar{w}W(w)^* + BH(w)^* = W(w)^* \quad (2)$$

and

$$C\bar{w}W(w)^* + DH(w)^* = I. \quad (3)$$

Since $\|V_{00}\|_{\text{op}} \leq \|V_0\|_{\text{op}} = 1$ because V_0 is an isometry, we have $\|A\|_{\text{op}} \leq 1$ and thus $\|A\bar{w}\|_{\text{op}} < 1$. Thus $I - A\bar{w}$ is invertible, and (2) yields

$$W(w)^* = (I - A\bar{w})^{-1}BH(w)^*.$$

Plugging this representation of $W(w)^*$ into (3) results in

$$C\bar{w}(I - A\bar{w})^{-1}BH(w)^* + DH(w)^* = I.$$

Taking adjoints and replacing w by z shows that

$$H(z) \left[D^* + B^*(I - zA^*)^{-1}zC^* \right] = I.$$

This right inverse is Schur by Lemma 3. ■

4.2 Minimum Norm Right Inverse

The extension of Theorem 12 from an upper bound one on right inverses to arbitrary bounds is a simple scaling argument. Note that in particular the upper bound $\gamma = \gamma_{\text{opt}}(H)$ is valid due to Corollary 6, and results in a minimum norm right inverse of H .

Corollary 13 *Let $\gamma_{\text{opt}}(H) \leq \gamma < \infty$. Denote by $\tilde{G} \in S(\mathcal{E}_*, \mathcal{E})$ the right inverse to $\tilde{H} := \gamma H$ as given by Theorem 12. Then $G := \gamma \tilde{G}$ is a right inverse of H with $\|G\|_\infty \leq \gamma$.*

PROOF. Since $\gamma_{\text{opt}}(H) \leq \gamma < \infty$, a right inverse $\check{G} \in H^\infty(\mathcal{E}_*, \mathcal{E})$ of H with $\|\check{G}\|_\infty \leq \gamma$ exists by Corollary 6. Thus

$$\gamma H \gamma^{-1} \check{G} = I, \quad \|\gamma^{-1} \check{G}\|_\infty \leq 1,$$

which shows that $\tilde{H} = \gamma H$ has a right inverse in $S(\mathcal{E}_*, \mathcal{E})$. Let $\tilde{G} \in S(\mathcal{E}_*, \mathcal{E})$ denote the right inverse of \tilde{H} given by Theorem 12. Then $G = \gamma \tilde{G}$ holds $\|G\|_\infty = \gamma \|\tilde{G}\|_\infty \leq \gamma$ as well as

$$HG = \gamma^{-1} \tilde{H} \gamma \tilde{G} = I.$$

■

5 Conclusions

In this paper we considered the problem of the construction of a causal precoder with optimal robustness for a stable and causal LTI system with multiple inputs and outputs. This problem is equivalent to finding a solution to the Bezout Identity with minimized peak value, for which we gave an explicit construction. We derived a novel method for numerical computation of the lowest peak value achievable in this problem, because it has to be known prior to the construction of the optimal precoder. This method is based on computation of a singular value decomposition of the finite section of a certain infinite block Toeplitz matrix, which is directly constructed from the Fourier coefficients of the systems transfer function.

Appendix

The complete proof of Theorem 7 follows splitted in three propositions.

The first proposition shows that $\{\rho_N(H)\}_N$ is monotonically decreasing and converges with a limit not lower than $\rho(H)$.

Proposition 14 *It holds*

$$\rho_N(H) \geq \rho_{N+1}(H) \geq \rho(H)$$

for all $N \in \mathbb{N}$.

PROOF. Let $u \in H^2(\mathcal{E}_*)$. We set $v := P_N u$ and $w := T_{H^*} v$. A simple computation shows that the Fourier coefficients of $w = P_+(H^* v)$ are given by

$$\hat{w}_k = \begin{cases} \sum_{j=0}^{\infty} \hat{H}_j^* \hat{v}_{k+j}, & k \geq 0 \\ 0, & k < 0 \end{cases}.$$

Since by construction $\hat{v}_k = 0$ for $k > N$, we see that $\hat{w}_k = 0$ for $k > N$. Thus

$$\|T_{H^*} P_N u\|_2^2 = \|w\|_2^2 = \sum_{k=0}^{\infty} \|\hat{w}_k\|_2^2 = \sum_{k=0}^N \|\hat{w}_k\|_2^2 = \|P_N w\|_2^2 = \|P_N T_{H^*} P_N u\|_2^2 \quad (4)$$

holds by Parseval's Relation for every $u \in H^2(\mathcal{E}_*)$.

Because trivially $P_N H^2(\mathcal{E}_*) \subset P_{N+1} H^2(\mathcal{E}_*) \subset H^2(\mathcal{E}_*)$, we obtain with (4), that

$$\begin{aligned} \rho_N(H) &= \inf_{u \in P_N H^2(\mathcal{E}_*), \|u\|_2=1} \|P_N T_{H^*} u\|_2 \\ &= \inf_{u \in P_N H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2 \\ &\geq \inf_{u \in P_{N+1} H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2 \quad (= \rho_{N+1}(H)) \\ &\geq \inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2 \\ &= \rho(H). \end{aligned}$$

■

We now ensure that the limit of $\{\rho_N(H)\}_N$ also is not greater than $\rho(H)$.

Proposition 15 *For every $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that*

$$\rho_N(H) \leq \rho(H) + \epsilon$$

for all $N > K$.

PROOF. We assume $H \neq 0$ since the case $H = 0$ is trivially true. Let $\epsilon > 0$ and choose $\tilde{u} \in H^2(\mathcal{E}_*)$ with $\|\tilde{u}\|_2 = 1$ such that

$$|\|T_{H^*} \tilde{u}\|_2 - \rho(H)| = \left| \|T_{H^*} \tilde{u}\|_2 - \inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2 \right| \leq \frac{\epsilon}{6}. \quad (5)$$

Since $\tilde{u} \in H^2(\mathcal{E}_*)$, $T_{H^*} \tilde{u} \in H^2(\mathcal{E})$ and $\|\tilde{u}\|_2 = 1$, Parseval's Relation shows that

$$\lim_{N \rightarrow \infty} \|P_N \tilde{u} - \tilde{u}\|_2 = \lim_{N \rightarrow \infty} \|P_N T_{H^*} \tilde{u} - T_{H^*} \tilde{u}\|_2 = 0, \quad \lim_{N \rightarrow \infty} \|P_N \tilde{u}\|_2 = 1.$$

Thus $K \in \mathbb{N}$ exists such that

$$\|P_N \check{u} - \check{u}\|_2 \leq \frac{\epsilon}{6} \|T_{H^*}\|_{\text{op}}^{-1}, \quad (6)$$

$$\|P_N T_{H^*} \check{u} - T_{H^*} \check{u}\|_2 \leq \frac{\epsilon}{6} \quad \text{and} \quad (7)$$

$$\|P_N \check{u}\|_2 \geq \frac{\rho(H) + \frac{\epsilon}{2}}{\rho(H) + \epsilon} \quad (8)$$

for all $N > K$.

Then for $N > K$ it follows that

$$\begin{aligned} \|P_N T_{H^*} P_N \check{u} - T_{H^*} \check{u}\|_2 &\leq \|P_N T_{H^*} (P_N \check{u} - \check{u})\|_2 + \|T_{H^*} \check{u} - P_N T_{H^*} \check{u}\|_2 \\ &\leq \underbrace{\|P_N T_{H^*}\|_{\text{op}}}_{\leq \|T_{H^*}\|_{\text{op}}} \underbrace{\|P_N \check{u} - \check{u}\|_2}_{\leq \epsilon/(6\|T_{H^*}\|_{\text{op}}) \text{ by (6)}} + \underbrace{\|T_{H^*} \check{u} - P_N T_{H^*} \check{u}\|_2}_{\leq \epsilon/6 \text{ by (7)}} \\ &\leq \frac{\epsilon}{3} \end{aligned} \quad (9)$$

and therefore

$$\begin{aligned} &\left| \|P_N T_{H^*} P_N \check{u}\| - \inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2 \right| \\ &\leq \underbrace{|\|P_N T_{H^*} P_N \check{u}\|_2 - \|T_{H^*} \check{u}\|_2|}_{\leq \epsilon/3 \text{ by (9)}} + \underbrace{\left| \|T_{H^*} \check{u}\|_2 - \inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2 \right|}_{\leq \epsilon/6 \text{ by (5)}} \\ &\leq \frac{\epsilon}{2}. \end{aligned}$$

We see that

$$\|P_N T_{H^*} P_N \check{u}\|_2 \leq \inf_{u \in H^2(\mathcal{E}_*), \|u\|_2=1} \|T_{H^*} u\|_2 + \frac{\epsilon}{2} = \rho(H) + \frac{\epsilon}{2}. \quad (10)$$

Since $\|P_N \check{u}\|_2 > 0$ for $N > K$ by (8), the sequence $\{\check{u}_N\}_{N>K}$ given by

$$\check{u}_N := \frac{P_N \check{u}}{\|P_N \check{u}\|_2} \in P_N H^2(\mathcal{E}_*)$$

is well-defined. We obtain the intended result

$$\begin{aligned}
\rho_N(H) &= \inf_{u \in P_N H^2(\mathcal{E}_*), \|u\|_2=1} \|P_N T_{H^*} u\|_2 \\
&\leq \|P_N T_{H^*} \check{u}_N\|_2 \\
&= \frac{\|P_N T_{H^*} P_N \check{u}\|_2}{\|P_N \check{u}\|_2} \\
&\stackrel{(\text{by (10)})}{\leq} \frac{\rho(H) + \frac{\epsilon}{2}}{\|P_N \check{u}\|_2} \\
&\stackrel{(\text{by (8)})}{\leq} \rho(H) + \epsilon
\end{aligned}$$

for all $N > K$. ■

We know now by the Propositions 14 and 15 that the sequence ρ_N converges to ρ for $N \rightarrow \infty$. However it is still unclear, how ρ_N can be computed explicitly. The next proposition gives a simple formula for the numerical computation of ρ_N .

Proposition 16 *Let $H \in H^\infty(\mathbb{C}^{m \times n})$ with $m \leq n$ and set*

$$\Gamma_{H,N} := \begin{bmatrix} \hat{H}_0^* & \hat{H}_1^* & \dots & \hat{H}_N^* \\ 0 & \hat{H}_0^* & \dots & \hat{H}_{N-1}^* \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \hat{H}_0^* \end{bmatrix} \in \mathbb{C}^{n(N+1) \times m(N+1)}.$$

Then $\rho_N(H) = \sigma_{\min}(\Gamma_{H,N})$.

PROOF. Let $USV^* = \Gamma_{H,N}$ denote a singular value decomposition of $\Gamma_{H,N}$ with singular values

$$\sigma_1 \geq \dots \geq \sigma_{m(N+1)} \geq 0.$$

Then $U \in \mathbb{C}^{n(N+1) \times n(N+1)}$ and $V \in \mathbb{C}^{m(N+1) \times m(N+1)}$ are unitary matrices and $S \in \mathbb{C}^{n(N+1) \times m(N+1)}$ is of the form

$$S = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_{m(N+1)} & \\ & & & \end{bmatrix}.$$

Let $u \in P_N H^2(\mathbb{C}^n)$ and set $v := P_N T_{H^*} u$. We saw already in the proof of

Proposition 14, that the non-zero Fourier coefficients of v are uniquely determined by the relation

$$\begin{bmatrix} \hat{v}_0 \\ \hat{v}_1 \\ \vdots \\ \hat{v}_N \end{bmatrix} = \begin{bmatrix} \hat{H}_0^* & \hat{H}_1^* & \dots & \hat{H}_N^* \\ 0 & \hat{H}_0^* & \dots & \hat{H}_{N-1}^* \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \hat{H}_0^* \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{bmatrix} = \Gamma_{H,N} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{bmatrix}.$$

Thus by Parseval's Relation

$$\begin{aligned} \rho_N(H) &= \inf_{u \in P_N H^2(\mathbb{C}^m), \|u\|_2=1} \|P_N T_{H^*} u\|_2 \\ &= \inf_{u \in \mathbb{C}^{m(N+1)}, \|u\|_2=1} \|\Gamma_{H,N} u\|_2 \\ &= \inf_{u \in \mathbb{C}^{m(N+1)}, \|u\|_2=1} \|S u\|_2 \\ &= \sigma_{m(N+1)} \\ &= \sigma_{\min}(\Gamma_{H,N}). \end{aligned}$$

■

References

- [1] D. Falconer, S. Ariyavisitakul, Broadband wireless using single carrier and frequency domain equalization, in: Proc. IEEE WPMC, 2002, pp. 27–36.
- [2] R. Fischer, J. Huber, Signal processing in receivers for communication over mimo isi channels, in: Proc. IEEE ISSPIT, 2003, pp. 298–301.
- [3] H. Myung, J. Lim, D. Goodman, Single carrier fdma for uplink wireless transmission, IEEE Veh. Technol. Mag. 1 (3) (2006) 30–38.
- [4] A. Scaglione, G. Giannakis, S. Barbarossa, Redundant filterbank precoders and equalizers part i: Unification and optimal designs, IEEE Trans. Signal Process. 47 (7) (1999) 1988–2006.
- [5] S. Kung, Y. Wu, X. Zhang, Bezout space-time precoders and equalizers for mimo channels, IEEE Trans. Signal Process. 50 (10) (2002) 2499–2514.
- [6] G. Gu, L. Li, Worst-case design for optimal channel equalization in filterbank transceivers, IEEE Trans. Signal Process. 51 (9) (2003) 2424–2435.
- [7] G. Gu, E. Badran, Optimal design for channel equalization via the filterbank approach, IEEE Trans. Signal Process. 52 (2) (2004) 536–545.

- [8] S. Wahls, H. Boche, Stable and causal lti-precoders and equalizers for mimo-isi channels with optimal robustness properties, in: Proc. International ITG/IEEE Workshop on Smart Antennas (WSA), Vienna, Austria, 2007.
- [9] H. Boche, V. Pohl, Mimo-isi channel equalization – which price we have to pay for causality, in: Proc. EUSIPCO, Florence, Italy, 2006.
- [10] L. Li, G. Gu, Design of optimal zero-forcing precoders for mimo channels via optimal full information control, IEEE Trans. Signal Process. 53 (8) (2005) 3238–3246.
- [11] K. Zhou, J. Doyle, K. Glover, Robust and Optimal Control, Prentice-Hall, Upper Saddle River, NJ, 1996.
- [12] T. Trent, An algorithm for corona solutions on $h^\infty(d)$, Integral Equations and Operator Theory 59 (3) (2007) 421–435.
- [13] N. Vucic, H. Boche, Equalization for mimo isi systems using channel inversion under causality, stability and robustness constraints, in: Proc. IEEE ICASSP, Toulouse, France, 2006.
- [14] N. Nikolski, Operators, Functions, and Systems: An Easy Reading Vol. 1, Vol. 92 of American Surveys and Monographs, American Mathematical Society, Providence, 2002.
- [15] H. Boche, V. Pohl, General structure of the causal and stable inverses of mimo systems with isi, in: Proc. IEEE PIMRC, Berlin, Germany, 2005, pp. 102–106.
- [16] B. Sz.-Nagy, C. Foias, Harmonic Analysis of Operators On Hilbert Spaces, North-Holland Publishing, London, 1970.
- [17] J. Ball, T. Trent, Unitary colligations, reproducing kernel hilbert spaces and nevanlinna-pick interpolation in several variables, J. Functional Analysis 157 (1998) 1–61.
- [18] M. Brodskii, Unitary operator colligations and their characteristic functions, Russian Math. Surveys 33 (1978) 159–191.
- [19] V. Tolokonnikov, Estimates in the carleson corona theorem, ideals of the algebra h^∞ , a problem of sz.-nagy, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 113.
- [20] T. Georgiou, M. Smith, Optimal robustness in the gap metric, IEEE Trans. Autom. Control 35 (6) (1990) 673–686.
- [21] J. Agler, J. McCarthy, Nevanlinna-pick interpolation on the bidisk, Journal für die reine und angewandte Mathematik 506 (1999) 191–204.